PRIME MAPPINGS

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1. **Introduction.** Suppose $A_n = \{a_1, a_2, \ldots, a_{2n}\}$ is a set of $2n$ points lying in the open interval (0, 1) such that $a_i < a_{i+1}$, $i = 1, \ldots, 2n-1$ and that W is a decomposition of A_n into two element sets. Suppose also that f is a mapping of the half open interval [0, 1) into the plane such that (1) $f(t) = f(t')$ for $t < t'$ if and only if $\{t, t'\} \in W$, (2) Im f can be expressed as the sum of a finite number of straight line intervals such that no point of $f(A_n)$ is an endpoint of one of the intervals and, (3) $f(t) \to f(0)$ as $t \to 1$. The decomposition W is said to determine the double point structure of f, and W is said to have property P provided it is true that if U and V are subsets of W such that $U = W - V$, then there exist $\{u_1, u_2\} \in U$ and $\{v_1, v_2\} \in V$ such that $u_1 < v_1 < u_2 < v_2$ or $v_1 < u_1 < v_2 < u_2$. If W has property P and the double point structure of f is determined by W then f is said to have property P or be prime. It is now possible to state two of the main results.

THEOREM 2 (THE INVARIANCE OF BOUNDARY THEOREM). *If A_n (as above) is a set of $2n$ points lying in (0, 1), W is a decomposition of A_n into two element sets and f and g are prime mappings whose double point structure is determined by W, then there is a natural one-to-one correspondence between the complementary domains of* Im f *and those of* Im g *according to the equation* $f^{-1}(\text{Bd } U) = g^{-1}(\text{Bd } V)$, *where U and V are corresponding complementary domains of* Im f *and* Im g, *respectively.*

THEOREM 3. *Given A_n, W, f and g as in Theorem 2, and assuming that the unbounded complementary domains of* Im f *and* Im g *correspond, then there is a homeomorphism h from E^2 onto E^2 such that $hf = g$.*

The main use of Theorem 3 (to the author), and, certainly the context in which it arose, are now described. Given f as in paragraph one, the set Im f can be considered [1] as the projection of a polygonal knot in regular position, where the set $f(A_n)$ is the set of double points of the projection. Suppose g is a one-to-one mapping of [0, 1) into E^3 so that (1) $\pi g = f$, where $\pi(x, y, z) = (x, y, 0)$, (2) Im g is the sum of a finite number of straight line intervals, and (3) $g(t) \to g(0)$ as $t \to 1$. D. E. Penney [6] has been studying the idea of associating with g (or Im g) a "word" $f(a_1)^{e_1} f(a_2)^{e_2} \cdots f(a_{2n})^{e_{2n}}$, where if $f(a_i) = f(a_j)$ and the z coordinate of $g(a_i)$ is larger than the z coordinate of $g(a_j)$, then $e_i = 1$ (or is suppressed) and $e_j = -1$. The technique is illustrated in Figure 1.

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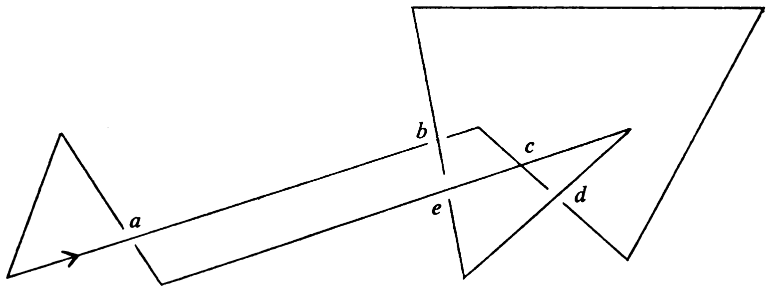
In [6] Penney describes a set of "admissible" operations with "words", one example of which is the cancellation of the aa^{-1} in the "word" associated with Figure 1. His Theorem 1 (applied here) says that there is some knot isomorphic to the one of Figure 1 and whose word is $b^{-1}cd^{-1}be^{-1}dc^{-1}e$. A prime word is one where πg is a prime mapping. Penney's Theorem 3 says that if F and G are knots



$ab^{-1}cd^{-1}be^{-1}dc^{-1}ea^{-1}$

FIGURE 1

with words W_1 and W_2, respectively, W_2 is prime and can be obtained by a finite number of "admissible" operations on W_1, then F and G are isomorphic. Theorem 3 of this paper is one of the preliminaries to the Theorem 3 of Penney's paper.

For other references in the field of topology see Gauss [2], Nagy [5] and Treybig [9]. For references in the field of topological analysis see Marx [3] and Titus [7] and [8].

2. **Definitions.** In addition to the definitions stated in the introduction it is desirable to state a few others. Given A_n a subset of (0, 1) as above, let $N(A_n)$ denote the set of all mappings f of [0, 1) into the plane such that there is a decomposition W of A_n into two element sets and such that f and W are related as above. Let $G(A_n)$ denote $\{[a_1, a_2], \ldots, [a_{2n-1}, a_{2n}], [0, a_1]+[a_{2n}, 1)\}$. The notation $[0, a_1]+[a_{2n}, 0)$ will be shortened to $[a_{2n}, a_1]$. As in the case of other intervals, the points a_{2n} and a_1 will be called the endpoints of this set. Given a collection H of point sets let H^* denote the sum (or union) of the sets in H. Given a decomposition W of A_n as above, then W will be said to have property Q provided it is true that if $\{a_i, a_j\} \in W$ then there exists $\{a_r, a_s\} \in W$ such that $i<r<j<s$ or $r<i<s<j$. If W determines the double point structure of $f \in N(A_n)$ and W has property Q, then f will be said to have property Q. It is easy to see that property P implies property Q for $n>1$.

3. **A lemma.**

LEMMA 1. *If $f \in N(A_n)$ and each of AB and CD is an arc lying in* Im f *such that* $\{A, B, C, D\} \subset f(A_n)$, *then* (1) *there exists H, K such that $H \subset G(A_n)$, $K \subset G(A_n)$, $f(H^*)=AB$ and $f(K^*)=CD$, and* (2) *if AB and CD intersect, then the first* (*last*) *point of CD on AB in the order from A to B is in $f(A_n)$.*

Proof. (1) Let $H=\{G\in G(A_n): f(G-G\cdot A_n)$ intersects $AB\}$. If $f(G-G\cdot A_n)$ intersects AB then it must be a subset of arc AB or it would contain one of the end-points A or B. Therefore, since AB is closed, H contains a finite number of closed sets and $f(H^*)$ is dense in AB, it follows that $f(H^*)=AB$. Define K for CD analogously.

(2) Suppose the first point X of CD on AB in the order from A to B is not in $f(A_n)$. There exists $G\in G(A_n)$ so that $X\in f(G-G\cdot A_n)$ so $G\in H\cdot K$. Thus there is an open interval containing X which is a subset of both AB and CD. This yields a contradiction.

4. The theorems.

THEOREM 1. *If A_n (as above) is a set of $2n$ points lying in $(0, 1)$, W is a decomposition of A_n into two element sets and f and g are prime mappings whose double point structure is determined by W, then for each complementary domain U of* Im f *there is a unique complementary domain V of* Im g *such that $f^{-1}(\text{Bd } U)=g^{-1}(\text{Bd } V)$.*

Proof. By Theorem 7 of [9] the collection H_1 of all elements K of $G(A_n)$ such that $f(K-K\cdot A_n)$ intersects Bd U has the property that $f(H_1^*)=\text{Bd } U$, and is the only subcollection of $G(A_n)$ with this property. Suppose $W_1=\{a_r, a_s\}\in W$ $(r<s)$ and that each of I_1 and I_2 is an element of H_1 such that a_r is an endpoint of I_1 and a_s is an endpoint of I_2. (see Theorem 9 of [9]). By Theorem 5 of [9] there is a complementary domain V of Im g such that $g(I_1)+g(I_2)\subset \text{Bd } V$. Let H_2 denote the unique subcollection of $G(A_n)$ such that $g(H_2^*)=\text{Bd } V$. The idea is to show that $H_2=H_1$, so suppose that $H_1\neq H_2$.

If $n=1$ then it follows that $H_1=H_2$, so the previous assumption means that $n>1$, and that W has property Q. By Theorem 9 of [9], (1) Bd U (Bd V) is a simple closed curve, and (2) if $a_i\in A_n\cdot K\subset K\in H_1$ (H_2) there is exactly one other element L of H_1 (H_2) containing an element a_j of A_n such that $f(a_i)=f(a_j)$, and furthermore K and L do not intersect. With the aid of Lemma 1 H_1 (H_2) can be expressed as the sum of two subcollections $\{I_2\}$ and $\{J_1, J_2, \ldots, J_{m_1}\}$ ($\{I_2\}$ and $\{K_1, \ldots, K_{m_2}\}$) such that (1) $f(I_2)$ and $f(\sum J_p)$ ($g(I_2)$ and $g(\sum K_p)$) are two arcs which meet only in their endpoints, and (2) $f(J_p)$ intersects $f(J_q)$ ($g(K_p)$ intersects $g(K_q)$) if and only if $|p-q|\leqq 1$, $1\leqq p, q\leqq m_1$ (m_2), (3) $K_1=J_1=I_1$. There is an integer n_1 such that $J_p=K_p$ for $1\leqq p\leqq n_1$, but $J_{n_1+1}\neq K_{n_1+1}$. Furthermore, there is an integer $n_2>n_1$ such that $f(J_{n_2})$ intersects $f(\sum K_p)$, but if $n_1<q<n_2$ then $f(J_q)$ does not intersect $f(\sum K_p)$.

As above H_1 (H_2) is the sum of two subcollections A_1 and A_2 (B_1 and B_2) such that (1) Bd U (Bd V) is the sum of two arcs $f(A_1^*)$ and $f(A_2^*)$ ($g(B_1^*)$ and $g(B_2^*)$) having only their endpoints in common, (2) $A_1=\{J_{n_1+1}, \ldots, J_{n_2}\}$, and (3) $f(B_1^*)$ has the same end points as $f(A_1^*)$ but they have no other point in common.

If $f(B_2^*)$ intersects $f(A_1^*)$ in a point distinct from one of its endpoints, then $f(B_2^*)$ contains an arc D_2 and $f(B_1^*)$ contains an arc D_1 such that (1) D_1 and D_2 do not intersect, (2) D_1 and D_2 lie except for their endpoints in the complement of $\overline{U}$ and the endpoints of D_1 separate those of D_2 on Bd U. By Theorem 11, p. 147 of [4], D_1 and D_2 must intersect, which is a contradiction. Therefore, the situation is that (1) $f(A_1^*)\cdot f(A_2^*)=P+Q=f(B_1^*)\cdot f(B_2^*)$, (2) $f(H_2^*)\cdot f(A_2^*)=P+Q$ and (3) $f(H_1^*)\cdot f(B_2^*)=P+Q$, where $P=f(a_i)$, $Q=f(a_p)$, $\{a_i, a_j\}\in W$, $\{a_p, a_q\}\in W$, and a_i is an endpoint of $J_{n_1}=K_{n_1}$. It may be supposed without loss of generality that $i<j, p, q$ and that a_p is an endpoint of J_{n_2}. Let K denote the collection of all elements of $G(A_n)$ which have none of a_i, a_j, a_p and a_q for an endpoint. The idea now is to try to obtain a subset L of K such that $f(L^*)$ is connected and intersects both of $f(A_1^*)$ and $f(A_2^*)$, $f(A_1^*)$ and $f(B_2^*)$, $f(A_2^*)$ and $f(B_1^*)$, or $f(B_1^*)$ and $f(B_2^*)$.

Suppose that $J_{n_1}=K_{n_1}=[a_{i-1}, a_i]$, and note that one of $[a_{j-1}, a_j]$ and $[a_j, a_{j+1}]$ is an element of A_2 and the other is an element of B_2.

Case 1. $i<p$, $q<j$. Let $L=\{[a_1, a_2], \ldots, [a_{i-2}, a_{i-1}], [a_{j+1}, a_{j+2}], \ldots, [a_{2n}, a_1]\}$. ($i<p$, $q<j$ here means $i<p<j$ and $i<q<j$.)

Case 2. $i<q, j<p$ and $[a_p, a_{p+1}]$ is an element of A_2 or B_2. Let

$$L = [a_1, a_2], \ldots, [a_{i-2}, a_{i-1}], [a_{p+1}, a_{p+2}], \ldots, [a_{2n}, a_1].$$

Case 3. $i<p, j<q$ and $[a_q, a_{q+1}]$ is an element of A_2 or of B_2 (analogous to 2).

Case 4. $i<j, p<q$ and $[a_q, a_{q+1}]$ is not an element of A_2+B_2.

(a) $j<p$. In this case each of the sets

$$L_1 = \{[a_{i+1}, a_{i+2}], \ldots, [a_{j-2}, a_{j-1}]\},$$
$$L_2 = \{[a_{j+1}, a_{j+2}], \ldots, [a_{p-2}, a_{p-1}]\},$$

and

$$L_3 = \{[a_{p+1}, a_{p+2}], \ldots, [a_{q-2}, a_{q-1}]\}$$

has the property that $f(L_i^*)$ is connected ($i=1, 2, 3$) and intersects $f(A_2^*)$ or $f(B_2^*)$. Since W has property Q there is an element $\{a_v, a_w\}$ of W such that $i<v<q$ and $w<i$ or $w>q$. Let $L=L_t+\{[a_{q+1}, a_{q+2}], \ldots, [a_{2n}, a_1], [a_1, a_2], \ldots, [a_{i-2}, a_{i-1}]\}$ where a_v is an endpoint of an element of L_t.

(b) $p<j$. Let

$$L_1 = \{[a_{i+1}, a_{i+2}], \ldots, [a_{p-2}, a_{p-1}]\},$$
$$L_2 = \{[a_{p+1}, a_{p+2}], \ldots, [a_{j-2}, a_{j-1}]\}$$

and

$$L_3 = \{[a_{j+1}, a_{j+2}], \ldots, [a_{q-2}, a_{q-1}]\}.$$

If each of $f(U_1^*)$, $f(U_2^*)$ and $f(U_3^*)$ intersects one of $f(A_2^*)$ and $f(B_2^*)$, then proceed as in 4(a). If not, then $f(U_1^*)$ must be the one that fails to meet $f(A_2^*)$ or $f(B_2^*)$. Since f is prime there exists $\{a_v, a_w\}\in W$ such that $i<v<q$ and $w<i$ or $q<w$. If $p<v<j$ or $j<v<q$ proceed as in 4(a). Now suppose $i<v<p$. By condition 1 of Theorem 19 of [9] there is an element $\{a_t, a_u\}$ of W such that $u\neq j$, $p<u<q$, and $t<p$ or $t<q$. If $t>q$ or $t<i$ proceed as in 4(a). If $i<t<p$ let

$$L = L_1+L_m+\{[a_{q+1}, a_{q+2}], \ldots, [a_{2n}, a_1], [a_1, a_2], \ldots, [a_{i-2}, a_{i-1}]\}$$

where a_u is an endpoint of an element of L_m. This concludes cases 1–4 and shows that in any event the desired collection L is obtained.

Suppose $f(L^*)$ intersects $f(A_2^*)$ and $f(A_1^*)$. There is an arc AB from point A in $f(A_1^*)\cdot f(A_n)$ to a point B in $f(A_2^*)\cdot f(A_n)$. There is a subarc CD of AB such that (1) C and D are in $f(A_n)$, and (2) C belongs to one of $f(A_1^*)$ and $f(A_2^*)$ and D belongs to the other and arc CD misses $f(B_2^*)$ (or use $f(A_1^*)$ and $f(B_2^*)$ and require that CD miss $f(A_2^*)$, or use $f(A_2^*)$ and $f(B_1^*)$ and require that CD miss $f(B_2^*)$, or use $f(B_2^*)$ and $f(B_1^*)$ and require that CD miss $f(A_2^*)$) and (3) CD contains no proper subarc with the same property. Let $M=\{L' \in L : f(L') \subset CD\}$. (It follows by Lemma 1 that CD is the sum of such sets.)

Suppose for example that $C \in f(A_1^*)$ and $D \in f(A_2^*)$. The arc CD and the arc $f(B_1^*)$ have endpoints that separate each other on Bd U. Therefore CD and $f(B_1^*)$ intersect, a contradiction.

If $C \in f(A_1^*)$ and $D \in f(B_2^*)$ let R denote the first point of $f(B_1^*)$ in the order QCP on $f(A_1^*)$. Let N denote $\{L' \in L : f(L') \subset \text{arc } CR\}$. In this case $g(M^*)+g(N^*)$ and $g(A_2^*)$ are arcs whose endpoints separate each other on Bd V. This involves a contradiction.

These two cases suffice to show how to handle the other two, so this concludes the proof of Theorem 1.

Proof of Theorem 2. For each complementary domain U of Im f, let U' denote the unique complementary domain of Im g guaranteed by Theorem 1. But starting with Im g and letting a U' correspond to U'' one must obtain the relation

$$U \to U' \to U'' = U.$$

Proof of Theorem 3. Suppose the complementary domain U of Im f corresponds to the complementary domain U' of Im g under the correspondence guaranteed by Theorem 2. Let $B(U)$ denote the subcollection of $G(A_n)$ such that $f(B(U)^*)=\text{Bd } U$. Of course, $g(B(U)^*)=\text{Bd } U'$. Also for B_1, $B_2 \in B(U)$ $f(B_1)$ intersects $f(B_2)$ if and only if $g(B_1)$ intersects $g(B_2)$. Remember also that for $\{a_i, a_j\} \in W$, $f(a_i)=f(a_j)$ and $g(a_i)=g(a_j)$.

By the Schoenflies Theorem there exist homeomorphisms h_1 and h_2 mapping $\bar{U}$ and $\bar{U}'$, respectively, onto $T=\{z : |z| \leqq 1\}$ (for the case of the unbounded components use $T-\{0\}$). Define h mapping $\bar{U}$ onto $\bar{U}'$ by $h(h_1^{-1}(0))=h_2^{-1}(0)$ and otherwise, for $P \in W-\{0\}$ let t be a number in $[0, 1)$ such that $P=rh_1(f(t))$ for some r satisfying $0<r\leqq 1$ and define $h(h_1^{-1}(P))$ to be $h_2^{-1}(rh_2(g(t)))$. Let h be defined on all other $\bar{U}$'s analogously. It is a simple matter to check that h is a homeomorphism from E^2 onto E^2 such that $hf=g$. This concludes the proof of Theorem 3.

The collection $N(A_n)$ could also be defined for mappings into the two spheres, where the crossing of straight line intervals is replaced by the crossing of arcs (see [4, p. 182]). In this case Theorem 3 holds unrestricted in the sense that boundedness requirements simply disappear.

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